Developing Mathematical Habits of Mind

By Annie Selden and Kien Lim
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A Project NeXT panel, "Helping Students Develop Mathematical Habits of Mind without Compromising Key Concepts from the Syllabus," was organized by Kristin Camenga, Houghton College, and Kien Lim, University of Texas at El Paso, at the San Diego Joint Mathematics Meetings. The panel addressed the tension between teaching all the mathematical concepts listed in a syllabus and incorporating opportunities for students to develop mathematical habits of mind.

Al Cuoco, Education Development Center, opened with some ideas for helping students develop mathematical habits of mind. These included: (1) being explicit about one's own thinking, (2) working on problems *with* students, (3) making thought experiments an integral part of one's courses, (4) providing concrete experiences before introducing formality, and (5) looking for habits of mind characteristic of various branches of mathematics. He presented various analytic and algebraic habits, with one of latter being the habit of seeking structural similarities. For example, the problem of finding a polynomial f that goes through (a_1,b_1), (a_2, b_2), … (a_r,b_r), is, by the Remainder Theorem, the same as finding a polynomial f that satisfies $f(x) =(x-a_1)q_1(x)+b_1$, $f(x) =(x-a_2)q_2(x)+b_2$, … $f(x) =(x-a_r)q_r(x)+b_r$, which, in turn, is the same as looking for a solution to the simultaneous set of congruences $f(x) \equiv b_1 (\text{mod } (x-a_1))$, $f(x) \equiv b_2 (\text{mod } (x-a_2))$, …, $f(x) \equiv b_r (\text{mod } (x-a_r))$. This "sameness" can be made precise, highlighting the deep structural similarity between $\mathbf{Z}$ and $\mathbf{Q}[x]$. Cuoco has a joint paper with Goldenberg and Mark on this topic. [Habits of Mind: An Organizing Principle for a Mathematics Curriculum, *Journal of Mathematical Behavior*, 15(4), 375-401.]

Hyman Bass, University of Michigan, continued by considering things mathematicians do, which have variously been called "mathematical habits of mind," "ways of thinking" including dispositions and sensibilities, and "practices." These practices include: (1) asking "natural" questions, (2) seeking patterns or structure, (3) consulting the literature and experts, (4) making connections, (5) using mathematical language with care and precision, (6) seeking and analyzing proofs, (7) generalizing, and (8) exercising aesthetic sensibility and taste. Bass claimed that instructors can cultivate these practices while treating the basic curriculum responsibly and that such instruction can, and must, start *very early*. The knowledge and skills demanded of a teacher are considerable, but they can be learned, with proper support. Bass concluded with a video of a third-grade class that was discovering, exploring, and speculating about even and odd numbers thereby illustrating how a teacher might encourage such habits.

Next Guershon Harel, University of California at San Diego, asked the question: What is mathematics? He emphasized that mathematics teaching should not appeal to gimmicks, entertainment, rewards, or punishment, but rather focus on the learner's *intellectual need* by fully utilizing humans' remarkable capacity to be puzzled. Nor should mathematics curricula compromise mathematical integrity, which is determined by *ways of understanding* and *ways of thinking* that have evolved over many centuries of mathematical practice. For Harel, a *way of understanding* is a particular *product* of a mental act carried out by an individual. For example, ¾ requires an act of interpretation that could include many possible ways of understanding (products); for example, three objects out of four objects, the sum ¼+¼ +¼, the measure of the quantity resulting from dividing 3 units into 4 equal parts, the solution of the equation $4x=3$,or the equivalence class $\{3n/4n \mid n \neq 0\}$. In contrast, a *way of thinking* is a characteristic of a mental act. For example one's interpretation of symbols might be characteristically inflexible or flexible, non-referential or referential. In answer to his initial question, Harel proposed that *mathematics* consist of two complementary subsets. The first subset is a collection, or a structure of structures, consisting of particular axioms, definitions, theorems, proofs, problems, and solutions -- the institutionalized *ways of understanding* in mathematics throughout history. The second subset consists of all *ways of thinking*, which characterize the mental acts whose products comprise the first set. Harel has a more detailed paper, "What is Mathematics?", that can be downloaded from his website: http://math.ucsd.edu/~harel/Research/1Publications.htm . It will also appear in R. B. Gold & R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics from the Perspective of Mathematicians,* to be published by the MAA.

Finally, Annie Selden, New Mexico State University, focused on two specific habits of mind and how one might encourage them without compromising the syllabus. The first was persistence, a habit which is widely applicable, even outside of mathematics. The second was writing "Let *x* be a (fixed, but arbitrary) number" into the proof of a universally quantified statement, a habit that is narrowly applicable. How might persistence be cultivated in one's class? Selden gave an example from her sophomore-level transition-to-proof courses. At every assessment, she gives both a take-home and an in-class exam. On the final take-home exam, she asks students to tell her what they have gotten out of the course, but not to give back a list of topics. One response she particularly liked was, "I've learned that I can wake up in the middle-of-the-night thinking about a math problem." She conjectures that this student's persistence resulted from having multiple opportunities and motivation to work on problems for a substantial period of time such as take-home tests doable over one week. For the second habit of mind, she presented a vignette of how Dr. K helps students develop the habit of taking a fixed, but arbitrary *x* in a proof where the definition reads *for all* $\varepsilon > 0$*, there is a* $\delta > 0$ *such that for all real numbers x …* , something students are reluctant to do, preferring instead to consider *all x* in the proof. Dr. K requires students to hand in only three proofs per week and meticulously grades just one. Students are allowed one additional week to rewrite and resubmit the proof for more points. The rewritten proof is invariably better, often incorporating Dr. K's suggestions, such as ending the proof with, "*Since x was arbitrary, we have now shown the theorem to be true for all x*." Selden again conjectured that multiple opportunities and motivation to consider a fixed, but arbitrary object, as well as writing a rational, were instrumental in getting students to adopt this habit of mind. Instilling good habits of mind takes time. She proposed that in some cases students must "just do it" and understanding often follows somewhat later.

The slides of all four presentations are available at http://www2.edc.org/cme/showcase.html.

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